Development of a fast algorithm for solving the unsteady incompressible Navier-Stokes equations

Christopher Cox[1], Chunlei Liang[2] and Michael W. Plesniak[3]

[1,2]Computational Aerodynamics & Hydrodynamics Laboratory, [1,3]Biofluid Dynamics Laboratory

Department of Mechanical & Aerospace Engineering

[1] PhD Candidate, [2] Assistant Professor, [3] Professor & Chairman

Motivation

In computational fluid dynamics (CFD), high-order (3rd and above) spatially accurate methods used to solve large scale problems require fast convergence. To address this need, the current implementation uses an implicit LU-SGS time-stepping scheme to accelerate convergence rate of unsteady 2D incompressible flows on unstructured meshes.

Governing Equations

Consider the unsteady incompressible Navier-Stokes equations with artificial compressibility (AC)

$$\frac{1}{\beta}\frac{\partial p}{\partial \tau} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \nabla\cdot\vec{V}\to 0$$

$$\frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial t} + \frac{\partial (u^2+p-\nu u_x)}{\partial x} + \frac{\partial (vu-\nu u_y)}{\partial y} = 0$$

$$\frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial t} + \frac{\partial (uv-\nu v_x)}{\partial x} + \frac{\partial (v^2+p-\nu v_y)}{\partial y} = 0$$

$$\frac{\partial U}{\partial \tau} + \frac{\partial \hat{U}}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

Pseudo Time Stepping
- implicit
- explicit

Physical Time Stepping
- 2nd order backward differencing

High-order Method
- flux reconstruction

- Non-linear lower-upper symmetric Gauss-Seidel backward Euler (LU-SGS)
- Third-order three-stage Runge-Kutta (RK33)

Implicit Time Stepping

Pros
(1) permits a large time step, Δt → quickly establish divergence-free velocity field
(2) utilizes advanced time-stepping techniques for solving hyperbolic/parabolic PDEs
(3) parallel processing and mesh deformation friendly to solve fluid-structure interaction problems

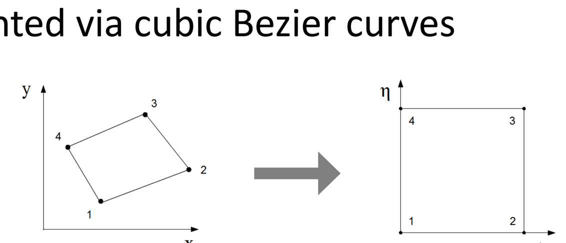
Cons
(4) high memory requirement & implementation difficulty

Mapping to Reference Element

We extend the idea of flux reconstruction[1,2] to solve incompressible flows with high order accuracy while implementing the following concepts for unstructured linear quadrilateral elements Ω_q:

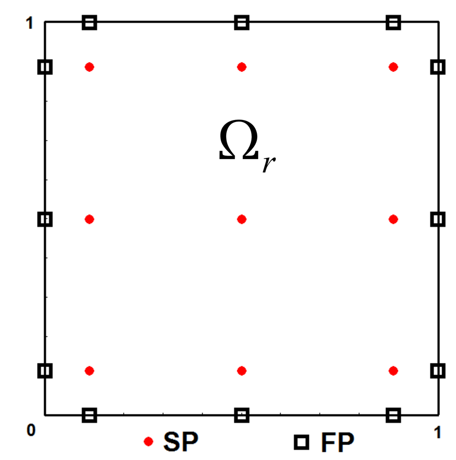
- isoparametric mapping of physical element Ω_q to reference element $\Omega_r = \{\xi,\eta \mid 0 \le \xi,\eta \le 1\}$
- curved boundaries represented via cubic Bezier curves

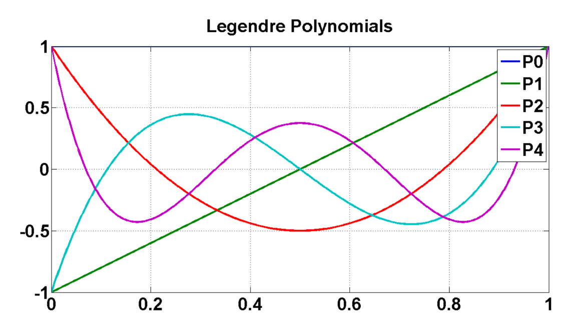
$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{i=1}^{4} \Psi_i(\xi,\eta)\begin{bmatrix} x_i \\ y_i \end{bmatrix}$$



High-order CFD Method

Each reference cell contains NxN solution points in 2D
- High order accuracy
- Solution points (SP) located at Legendre-Gauss positions
- Flux points (FP) located along the element boundary
- Solution is piecewise continuous across domain

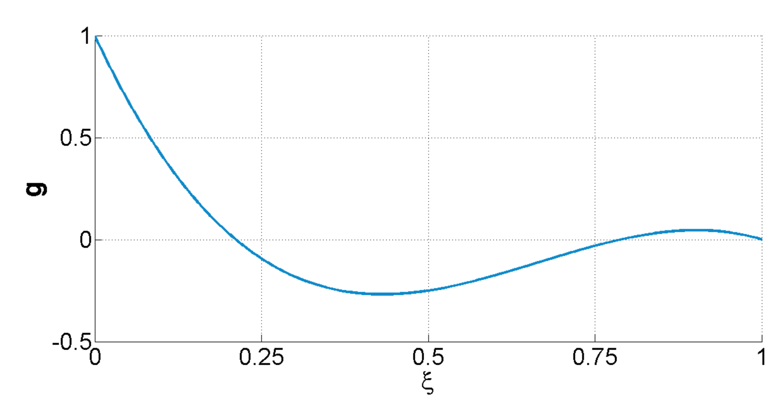




$$l_i(\xi) = \prod_{s=1,s\ne i}^{N}\left(\frac{\xi-\xi_s}{\xi_i-\xi_s}\right)$$

$$U(\xi) = \sum_{i=1}^{N} U_i l_i(\xi) \qquad f_r^D(\xi) = \sum_{i=1}^{N} f_{r|i}^D l_i(\xi)$$

$$f_r(\xi) = f_r^D(\xi) + [f_{r-1/2}^{com} - f_r^D(0)]g_r^{LB}(\xi) + [f_{r+1/2}^{com} - f_r^D(1)]g_r^{RB}(\xi)$$



Implicit Time-Stepping

Referring back to our mass conservation equation, we can define the residual for the first equation as

$$Residual: \; R_r = \nabla\cdot\vec{V}_r \to 0$$

and develop an algorithm to drive this residual as close and fast to zero as possible. To solve the governing form with the implicit LU-SGS scheme, a linearization of the governing equations must be performed.

$$\frac{p_r^{n+1,m+1} - p_r^{n+1,m}}{\Delta\tau} + \nabla\cdot\vec{V}_r^{n+1,m+1} = 0$$

$$\downarrow$$

$$R_r^{m+1} - R_r^m \approx \frac{\partial R_r}{\partial U_r} + \sum_{nb\ne r}\frac{\partial R_r}{\partial U_{nb}}\Delta U_{nb}$$

$$\downarrow$$

$$\left[\frac{I}{\Delta t} + \frac{\partial R_r}{\partial U_r}\right]\delta p_r^{k+1} = -R_r^* - \frac{\Delta p^*}{\Delta\tau}$$

$$\downarrow$$

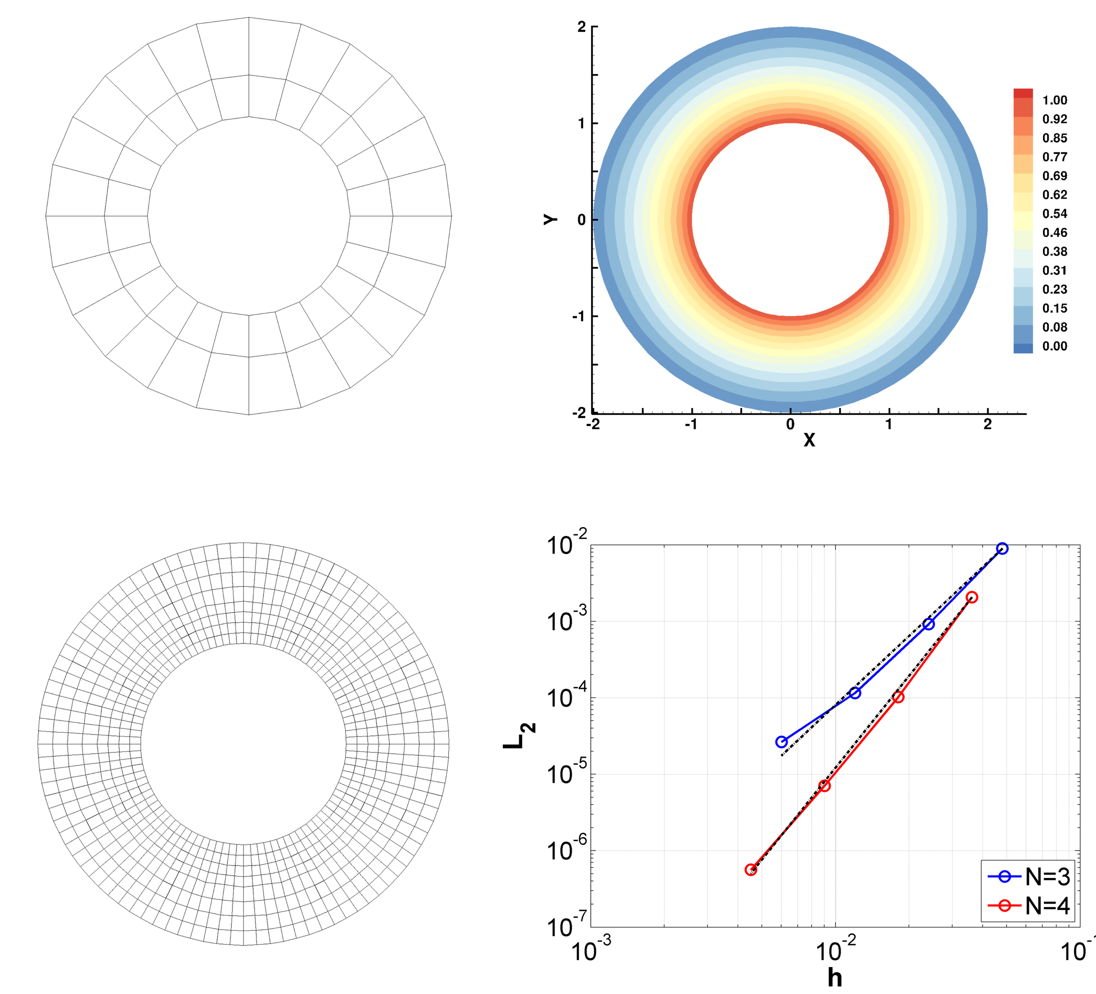
$$A \quad x \quad = \quad b$$

The system of equations is then solved directly using LU decomposition. For higher orders of accuracy, the size of matrix A renders the solution of x more computationally expensive.

N	A
2	12x12
3	27x27
4	48x48
5	75x75

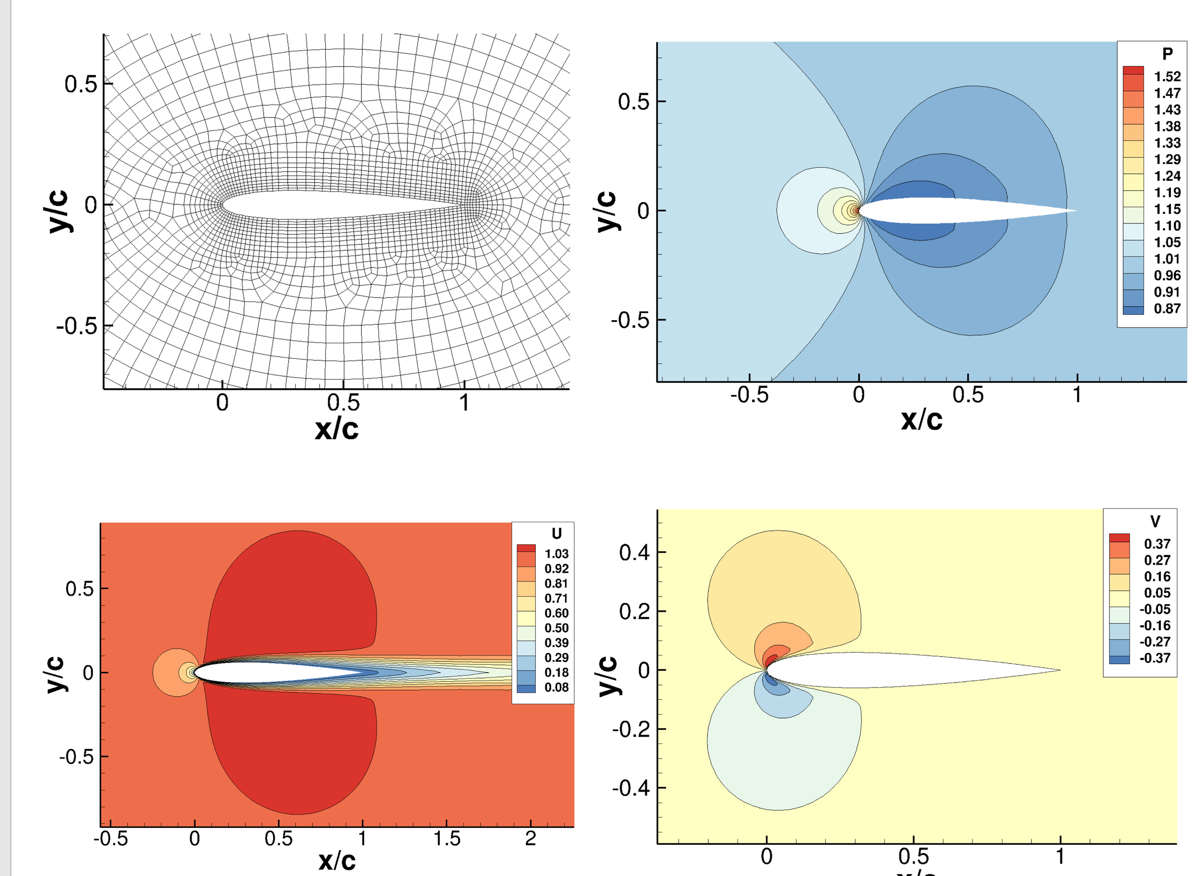
Verification

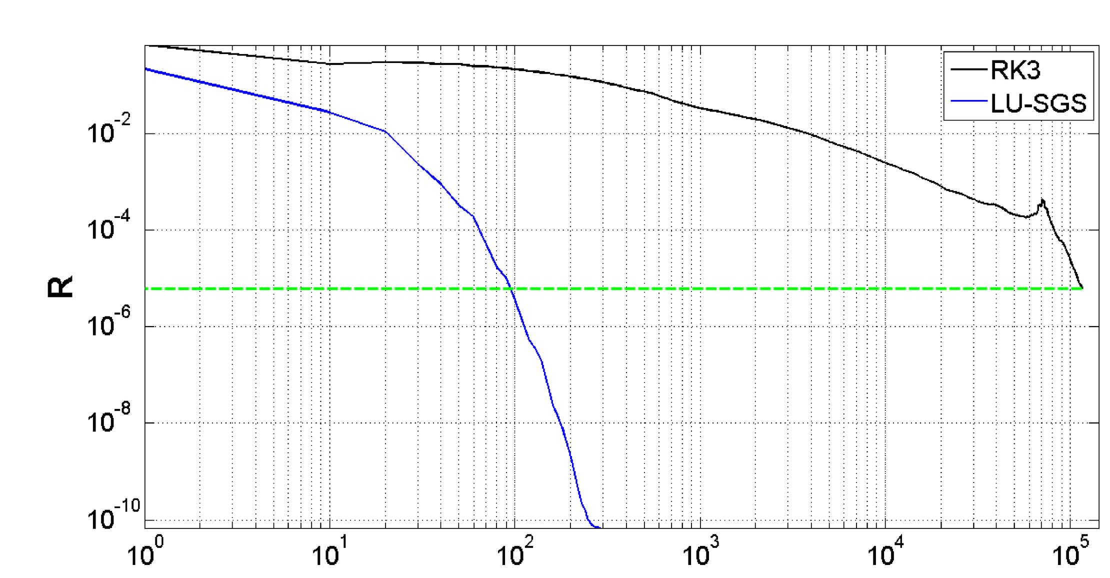
- Taylor-Couette Flow at *Re=10*



Steady Flow

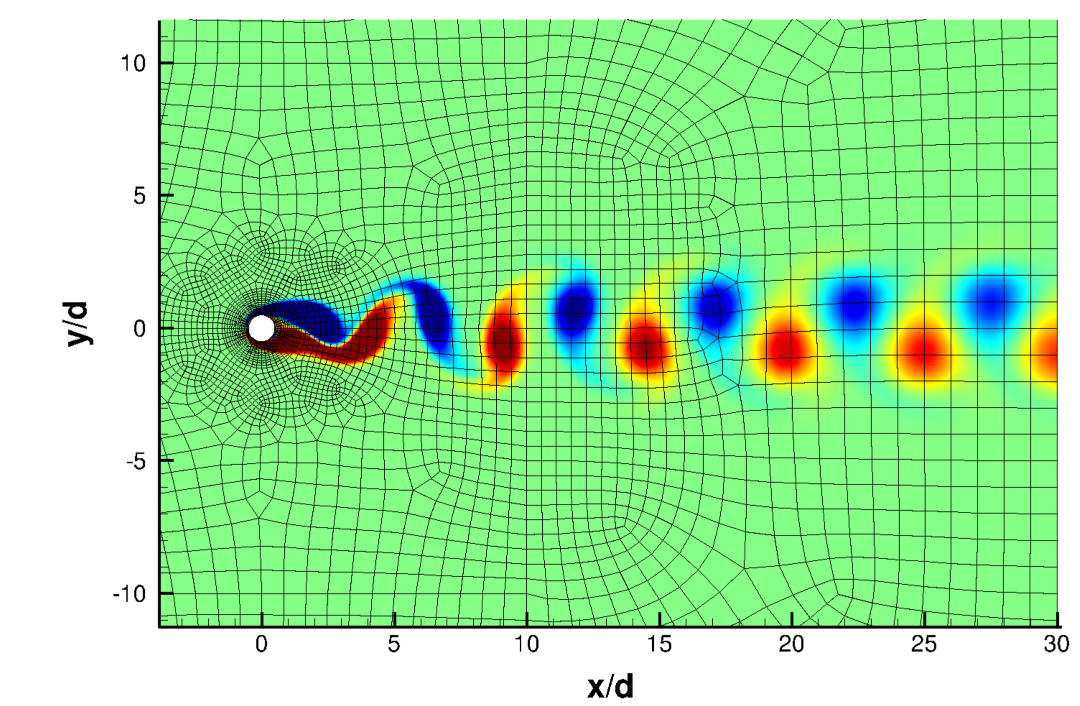
- NACA-0012 airfoil at *α=1°*, *Re=1850*
- Speedup in CPU time to achieve steady state *= 55*

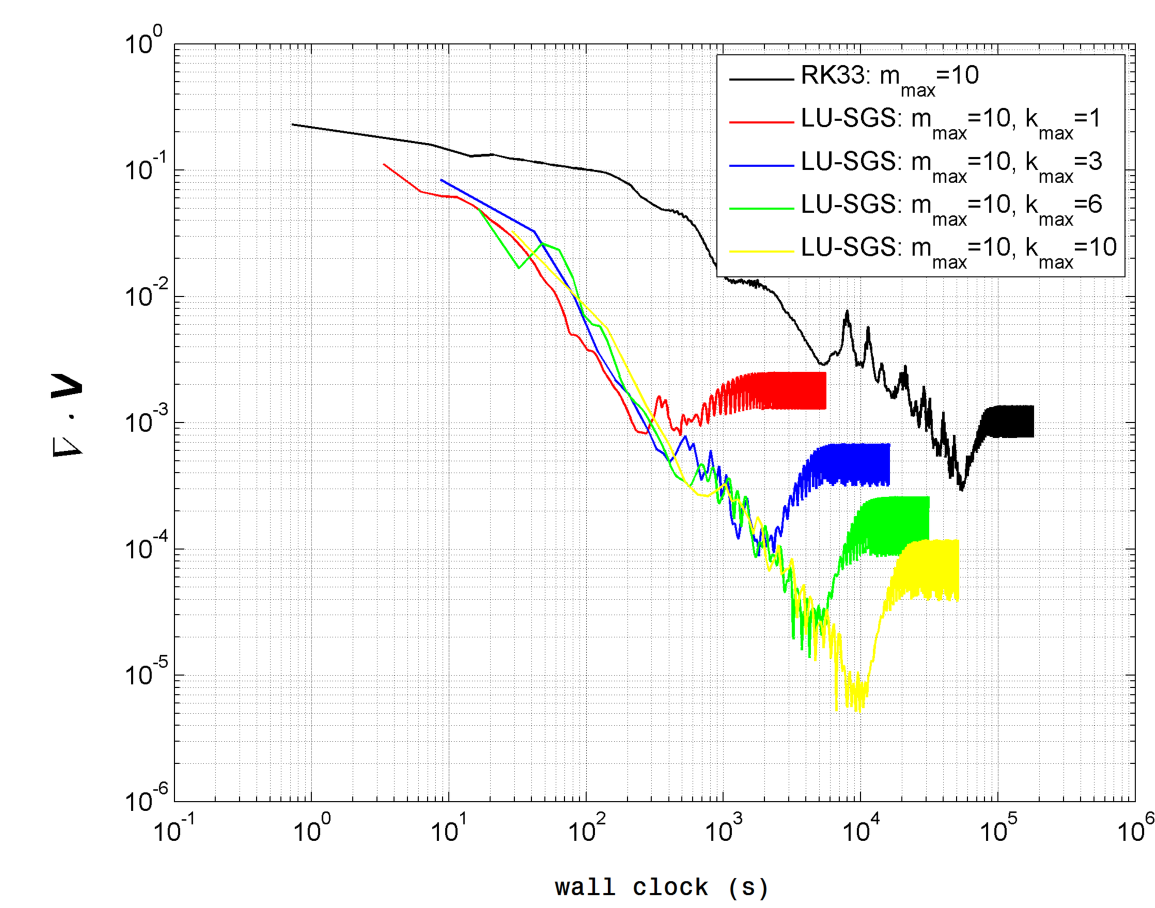




Unsteady Flow

- Cylinder at *Re=100*
- Efficiency speedup measured per shedding cycle





	Explicit	Implicit			
m_{max}	10	10	10	10	10
k_{max}	-	1	3	6	10
$C_{L,rms}$	0.231	0.240	0.230	0.223	0.222
C_D ($C_{D,rms}$)	1.354 (0.006)	1.342 (0.006)	1.338 (0.006)	1.338 (0.006)	1.338 (0.006)
Strouhal	0.162	0.160	0.162	0.163	0.163
Speedup	1	25.2	8.5	4.5	2.7

Future Implementation

(1) Parallel processing
(2) Three-dimensional implementation
(3) Moving and deforming mesh for fluid-structure interaction

References

[1] Huynh, H., "A flux reconstruction approach to high-order schemes including discontinuous Galerkin method," AIAA Paper AIAA-2007-4079, 2007.

[2] Wang, Z.J., Gao, H., "A unifying lifting collocation penalty formulation including the discontinuous Galerkin, spectral volume/difference methods for conservation laws on mixed grids," Journal of Computational Physics, **228**, 21, pp. 8161-8186, 2009.